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*Theory of Parallels.*

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THE PROOF OF

# EUCLID'S AXIOM

LOOKED FOR IN

THE PROPERTIES OF THE

# EQUIANGULAR SPIRAL.

---

*By Lieut. Colonel T. Perronet Thompson, F.R.S.*

OF QUEEN'S COLLEGE, CAMBRIDGE.

---

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# *Introduction.*

THE object in what follows is to demonstrate,

*That in the Equiangular Spiral the intercepted portion of the* radius vector *is of limited length for any limited number of revolutions, without dependence on the doctrine of parallels. And that any rectilinear series* [or number of equal straight lines in the same plane, joined one to another in succession by their extremities, and making at their several points of junction equal angles towards the same hand each less than the sum of two right angles] *can be confined by an Equiangular Spiral to which such series, however continued, shall always be interior. Whence the series shall rejoin itself, and the angular points lie in the circumference of a circle whose radius is limited and can be found.*

From this, either Euclid's Axiom, or the equality of the three angles of every triangle to two right angles, may be inferred in a variety of ways.

It will be objected, that the proposed demonstration is not elementary; or that it depends on the properties of a curve not found in the Elements of Euclid. But if the evidence is in the

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higher geometry of which Archimedes was the father, it is not to be rejected because the solution has not been brought within the narrower pale.

The grounds to be understood as previously established, consist of so much of the First Book of Euclid as precedes the appeal to the Twelfth Axiom; with the addition of any part of the doctrine of *proportion* from the Fifth Book, and of *variation* which is only the same in another dress. What innovations might hence in the end arise in the order of geometrical instruction, needs not be settled now. The want at present, is to get a clue of any kind and in any order, to the demonstration of the missing truth.

The first trial made, was with the spiral called "of Archimedes;" which would be effective, if it could be established (as upon view might be believed to be almost inevitable), that equal chords taken one after another beginning from the centre of revolution, contain greater and greater angles at their successive points of junction. But this property will be found dependent on the doctrine of parallels.

Such persons as may be acquainted with the earlier work entitled "*Geometry without Axioms*," are requested to consider the present publication as superseding what is to be found upon the same question in the former.

*T. Perronet Thompson.*

*London;* 21 *Old Buildings, Lincoln's Inn.* 11 *March,* 1840.

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# THEORY OF PARALLELS.

## NOMENCLATURES.

I. A magnitude is called *limited*, when its boundaries are assigned or could be. And the like of numbers.

By *unlimited* is meant that no boundaries are assigned at which the object in question shall be held to be necessarily terminated, but on the contrary it may without further notice be considered as continued to any extent a motive may ever arise for desiring*.

* See Note.

II. If a straight line revolves in any manner in a plane, about one of its extremities which remains at rest, and during the same time a point travels in any manner along the revolving straight line, the figure described by such point is called a *spiral*.

The revolving straight line with its prolongation to an unlimited extent, is called the *radius vector*, or for shortness the *radiant*; and the extremity about which it turns, is called the *centre of revolution*.

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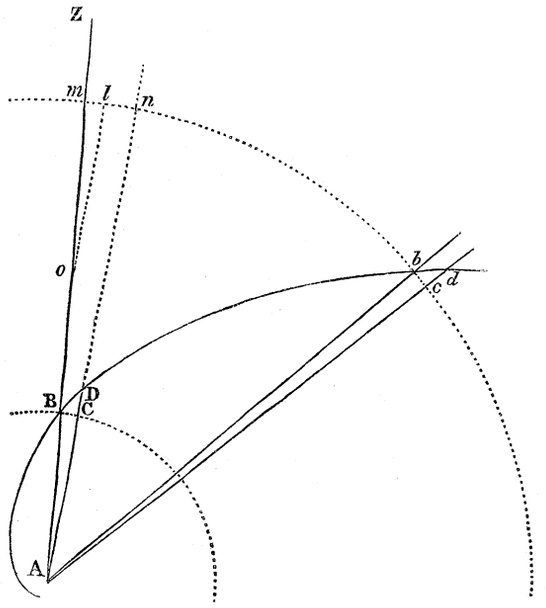
## PROPOSITION *A*.

PROBLEM.—*To describe a spiral such that the angle between the radiant and the curve at their intersection, shall in all parts of the figure be equal.*

While the radiant AZ revolves in any manner about A, let the travelling point be made to move so that its velocity along AZ at any point B in it, shall be always in a constant ratio to the velocity of B in the transverse direction given to it by the revolution of AZ. A spiral as required, shall be described.




With the radius AB describe a circle, and take BAD to represent a minute angle passed over by the radiant; whereupon BC and CD will represent the distances respectively described in one time, by B in the transverse direction, and by the travelling point along the radiant. At any other point in the spiral, as *b*, describe a circle with the radius A*b*, and take *bc* equal to BC, and let A*cd* be the radiant when it passes through *c*. Because BC and CD are described in one time, BC : CD : : velocity of B in the transverse direction : velocity at B along the radiant; and for the like reason, *bc* : *cd* : : velocity of *b* in the transverse direction : velocity at *b* along the radiant. But* the velocity of B in the trans-

* By the Hypothesis.

verse direction : velocity at B along the radiant : : velocity of *b* in the transverse direction : velocity at *b* along the radiant. Therefore BC : CD : : *bc* : *cd*; and by alternation, BC : *bc* : : CD : *cd*. But *bc* is* equal to BC; therefore *cd* is equal to CD. And because in the triangles *bcd*, BCD, the sides *bc*, *cd* are respectively equal to BC, CD, and the angle *bcd* equal to BCD (for they are right angles); the triangles are equal, and the angle *cdb* (or A*db*) is equal to the angle CDB (or ADB). And in like manner may be shown that the angle between the radiant and the curve at any other of their points of intersection will be equal to ADB.

* By Construction.

And by parity of reasoning, the like may be done in every other instance.

COROLLARY 1.—If the radiant revolves uniformly, the velocity of the travelling point along the radiant will be required to vary as the circumference of the circle whose radius is the distance of the travelling point from the centre of revolution.

For the velocity of any point B in the transverse direction, will be as the circumference of the circle whose radius is AB; because such circumferences are what would be described by the velocities in question in one time, viz. in the time of one complete revolution of the radiant. And because the velocity of the travelling point along the radiant is to be in a constant ratio to the velocity in the transverse direction, it is to be as the circumference aforesaid.

COR. 2.—During any† limited number of revolutions, the distance of the travelling point from the centre of revolution will be limited.

† Nomenclature I.

For as the travelling point recedes from the centre of revolution (the radiant revolving uniformly), at every limited distance from that centre its velocity will be limited, because the circumference of the circle described with a radius equal to such distance is‡ limited. And any limited velocity, whether uniform or varied, will in any limited time carry the point to a limited distance; for where the velocity is varied, the distance would have been limited if the greatest rate of velocity had been continued uniformly throughout the whole time, of necessity therefore will it be limited with the actual rates, some of which are less. Whence, during any limited time of revolution, the velocity and the distance will be mutually limited.

‡ See Note.

COR. 3.—If the motions are reversed; because the travelling

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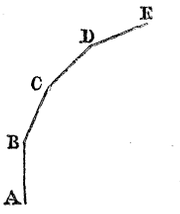
point, as it approaches the centre of revolution, will at some instant or instants move with velocities less than any which may be assigned, in the time of any limited number of revolutions it will not coincide with the centre of revolution.

Cor. 4.—The length of the curve described during any limited number of revolutions, will also be limited.

For if the triangle BAC were applied at $m$, so that B should be on $m$, and BA on $mo$, and AC fell on $ol$; because the angle $mol$ would be equal to BA$n$, $ol$ and A$n$ would be parallel, and consequently C fall on the side of $n$ which is towards $m$, and $mn$ be greater than BC. And because BD is less than the sum of CD and BC, still more is it less than the sum of CD and $mn$; therefore the sum of all the BD (which is B$b$), is less than the sum of all the CD (which is B$m$) and of all the $mn$ (which is $mb$). And because both B$m$ and $mb$ are limited, their sum is so, and so also is B$b$ which is less than their sum.

* See Note. Nomenclature 1.—A spiral as above, is called an *equiangular** spiral.

Nom. 2.—Equal straight lines in the same plane (as AB, BC, CD, &c.) joined one to another in succession by their extremities, and making at their several points of junction equal angles towards the same hand (as ABC, BCD, &c.) each less than the sum of two right angles, and so on for so far as it may be ever chosen to continue the same; are called a *rectilinear*† *series.*




† See Note.

The angle made by any two of the equal straight lines which are contiguous (that is to say, the angle ABC or any of its equals BCD, CDE, &c.) is called *the angle of the series.*

If a straight line of unlimited length is made to revolve about the first point A of the series, and so cut the different parts of the series in succession, it may be called the *radiant*, as in a spiral.

## PROPOSITION *B.*

Theorem.—*Any rectilinear series can be confined by an equiangular spiral to which such series, however continued, shall always be interior. And the distance from the first point in the series to any other shall be limited; and the number of the equal straight lines in the series up to such point, shall also be limited.*

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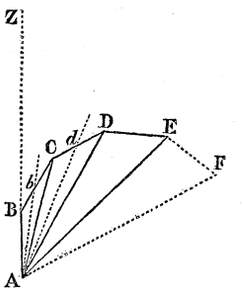
Let ABCDE&c. be a rectilinear series. And let a straight line AZ of unlimited length revolve about A in the manner of a radiant, beginning with passing through B, and passing successively through the other angular points C, D, E, &c. to whatever number it may be ever chosen to continue them.




When AZ in its revolution passes through C; because BA and BC are equal, the angle BCA will be equal to BAC. And when AZ passes through D; if the quadrilateral figure ABCD or its fac-simile were applied to itself by conversion, so that the points C, B of the one, should be on the points B, C of the other respectively, then by reason of the equality of the angles and straight lines respectively concerned, the figures would coincide, and the angle CDA be on the angle BAD, and coincide with it, and be equal to it; and in a similar manner may be shown that when AZ passes through E, it will make the angle DEA equal to BAE, and so on; and because the angle BAD is greater than BAC, and BAE than BAD, CDA which is equal to BAD is greater than BCA which is equal to BAC, and in like manner DEA than CDA, and so on. And when AZ passes through any point between B and C, as *b*, it will make an angle B*b*A greater than BCA, for it is the exterior angle of the triangle *b*CA and the other is one of the interior and opposite; still more when AZ passes through any point between C and D, as *d*, will it make an angle C*d*A greater than BCA, for the angle C*d*A is greater than CDA which is greater than BCA. Also so long as the inward angle made at the angular point last come to (as DEA, or its equal BAE) is less than the angle of the series, another of the equal straight lines of the series (as EF) will be lying on the outward side of the radiant, and a new triangle (as AEF) will be made on the further revolution of the radiant. And the portion of the radiant intercepted between A and F, cannot fail to be greater than between A and E, without the angle BAF having become greater than half the angle of the series; for if AF were *equal* to AE, the angle AFE would be equal to AEF, and it is greater than AED, therefore it would be greater than half DEF, and BAF which is equal to it would be greater also; and still more if AF were *less* than AE, for then the angle EFA would be greater than AEF as well as greater than AED.



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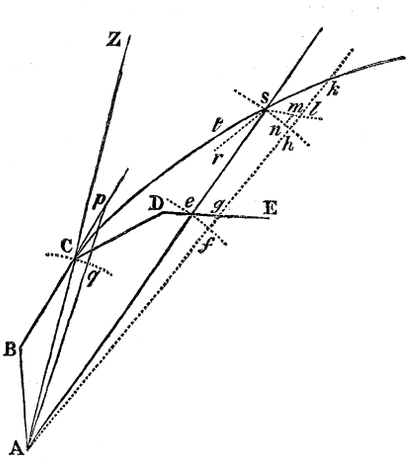



In the prolongation of BC (*See the Second figure opposite*) take now a minute distance C*p*, and join *p*A, and with the radius AC describe a circle cutting *p*A in *q*. And let a point, starting from C, travel along the radiant during its revolution, in such manner that its velocity along the radiant at any place in it as C, shall be to the velocity of C in the transverse direction, always as *qp* or some constant magnitude which is greater, to C*q*; whereupon such travelling point will* describe an equiangular spiral, in which the angle between the radiant and the curve at their point of intersection will always be equal to C*pq* (which is less than BCA, though by diminishing C*p* the difference can be made less than any magnitude which may be assigned), or to some constant angle smaller. And because the radiant on cutting the rectilinear series in any place as *e*, always makes the inward angle D*e*A greater than BCA, the series, however continued, shall always be interior to the spiral; for if any straight line of the series could approach and meet the spiral like *rs*, it must make with the radiant an angle *rs*A less (or at all events not greater) than *ts*A and consequently than BCA; which cannot be, for it has been shown always to make a greater. THEREFORE the portion A*e* of the radiant intercepted between A and the rectilinear series, will always be less than A*s* which is intercepted between A and the spiral; and because this last is† limited, the other is‡. AND FURTHER if the triangle *efg* (or *snm*) be applied upon the triangle *shk* so that the point *e* shall be on *s*, and the straight line *ef* on the straight line *sh*; because the angle *feg* (or *nsm*) is less than the angle *hsk* (for they are the respective complements of the angles D*e*A and *ts*A, of which D*e*A is the greater) the straight line *sl* will lie between *sh* and *sk*; and because *ef* as shown in Prop. *A*, Cor. 4, (or *sn*) is less than *sh*, and the angles *snm* and *shl* are right angles, *fg* (or *nm*) will be parallel to *hl*, and *eg* (or *sm*) will be less than *sl*; and because *skl* is an

* Prop. *A*.

† Prop. *A*. Cor. 2.

‡ See Note.

acute angle and *slk* an obtuse, *sl* which is opposite to *skl* is less than *sk* which is opposite to *slk*; still more then is *sm* (or *eg*) less than *sk*; therefore the sum of all the *eg* (or of the equal straight lines of the series, from C to the intersection of the series with the radiant at *g*) is less than the sum of all the *sk* (or than the length of the spiral, from C to its intersection with the radiant at *k*). And because this last is* limited, the other is; and because the sum of the equal straight lines in the series up to any point where it may be intersected by the radiant is limited, their number is.

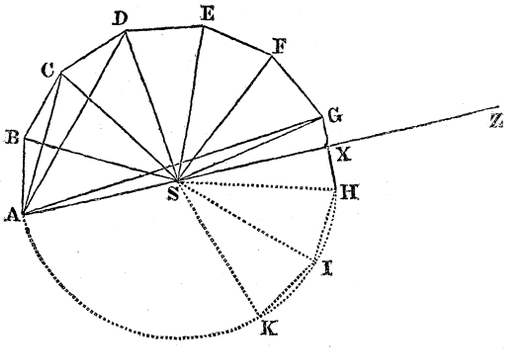
* Prop. *A.* Cor 4.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, any rectilinear series can be confined by an equiangular spiral &c. Which was to be demonstrated.

## PROPOSITION *C.*

THEOREM.—*Any rectilinear series, being continued, shall rejoin itself, and the angular points lie in the circumference of a circle whose radius is limited and can be found.*

Let ABCDE&c. be a rectilinear series. And let the radiant AZ be made to revolve till the angle BAX is equal to half the angle of the series, the series being continued by the successive addition of equal straight lines at its extremity as may be required;




whereupon has been shown† that the intercepted portion of the radiant will have become greater at every angular point successively passed through, and that the series will cut the radiant at a limited distance AX from A, the number of the equal straight lines which have been added to the series being also limited. Of the angular points of the series which are nearer the commencement of the series than X is, take the nearest to X, which is G; join AG, and from G draw GS bisecting the angle of the series FGH. Because the angle FGA is equal to BAG, which is

† Prop. *B.*



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less than BAX or than half the angle of the series, AGH is greater than half the angle of the series, and GS which bisects the angle of the series will lie between GA and GH, and being prolonged will cut the perimeter of the triangle AGX; and because it cannot cut it in GA or in GX (for then two straight lines would inclose a space), it will cut it in AX. Let it cut AX in S; and join SB, SC, SD, SE, SF. These as also SA and SG, shall be equal to one another; and a circle described with the centre S and radius SA, shall pass through the angular points B, C, D, E, F, G; and the other angular points of the series, if it be continued, shall lie in the circumference of the same circle.

For because the angle BAS is equal to FGS (inasmuch as they are each equal to half the angle of the series), and the angle BAG is equal to FGA, the remaining angle SAG is equal to SGA; wherefore in the triangle ASG, the side SG is equal to SA. If then SB be not also equal to SA, it must be either greater or less. And first let it be assumed that it is greater. But if SB be greater than SA, the angle SBA which is opposite to SA the less, must be less than SAB which is opposite to SB the greater; and SAB is* equal to half the angle of the series, or to half the angle ABC; therefore SBA must be less than half the angle ABC, and SBC greater than half, and consequently greater than SBA which is less than half. Therefore in the triangles SBC, SBA, because the sides SB, BC are equal to SB, BA respectively, but the angle SBC is greater than the angle SBA, the third side SC must be greater than the third side SA. Again, join AC; and because SC is greater than SA, the angle SCA must be less than the angle SAC. Add to each the equal angles BCA, BAC, and the whole angle SCB must be less than the whole angle SAB. But SAB is equal to half the angle of the series, or to half the angle BCD; therefore SCB must be less than half the angle BCD, and SCD greater than half, and consequently greater than SCB which is less than half. Therefore in the triangles SCD, SCB, because the sides SC, CD are equal to the sides SC, CB respectively, but the angle SCD is greater than SCB, the third side SD must be greater than the third side SB. But SB was greater than SA; still more, therefore, must SD be greater than SA. In like manner, by joining AD (whereupon the angle CDA will be equal to BAD), may be shown that SE must be greater than SC, and consequently than SB, and than SA; and so on with each of the other straight lines drawn from S to the angular points, in succession. Where-

* Constr.

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fore it would follow that SG must be greater than SA; which is impossible, for it has been shown to be equal to it. The assumption, therefore, which involves this impossible consequence, cannot be true; or SB is not greater than SA. And in the same way may be shown that it is not less. But because SB is neither greater than SA nor less, it is equal to it. And because SA, SB are equal, the angle SBA is equal to SAB; but SAB is* equal to half the angle of the series, or to half the angle ABC; therefore SBA is half the angle ABC, and consequently SBC is also half. And because SB, BC are equal to SA, AB respectively, and the angle SBC is equal to SAB (for they are each equal to half the angle of the series), the third side SC is equal to the third side SB, and the angle SCB is equal to the angle SBA, that is to say is half the angle of the series, and consequently the angle SCD is also half. And in the same way may be shown in succession, that SD, SE, SF, SG, and SH, are severally equal to one another and to SA, and that the angles of the series are severally bisected by the straight lines BS, CS, DS, ES, FS, and GS, and that SHG is equal to half the angle of the series. Therefore if with the centre S and radius SA a circle be described, by reason of the equality of SA, SB, &c. it will pass through all the angular points of the series which have been already formed, and through H. And if in the circumference of such circle be taken more equal straight lines in succession, as HI, IK, and as many more as it may be ever chosen to add, and from their several points of junction straight lines be drawn to S; because in the triangles SHI, SAB, the sides SH, SI are equal to SA, SB respectively, and the third side HI equal to the third side AB, the triangles are equal, and the angles SHI, SIH are equal to the angles SAB, SBA respectively, and consequently each equal to half the angle of the series. And in like manner in the triangle SIK, &c.; therefore the angles GHI, HIK, &c. are each equal to the angle of the series, and HI, IK, &c. which lie in the circumference of the circle, are the equal straight lines of the series which should be added in continuation, and none other. Wherefore the series being continued will rejoin itself, either in A or in some point between A and B†.

* Constr.

† See Note.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, any rectilinear series, being continued, shall rejoin itself, &c. Which was to be demonstrated.

Scholium.—The next Proposition contains the matter of Euclid's Twelfth Axiom, and is in fact only the opposite of the Theorem demonstrated by Euclid (I. 28.), that if two straight lines (in the same plane) are cut by a third, and the two interior angles on the same side of the cutting straight

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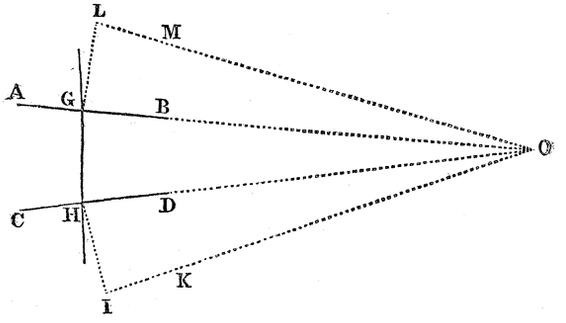
line *are* together equal to two right angles, the other two straight lines shall *never* meet; its object being to assert, that if the angles are *not* together equal to two right angles but less, though by ever so small a difference, the two straight lines *shall* meet.

The first of these two Propositions is commonly thought to be made clearer, by being put in the form of declaring that if *the alternate angles are equal*, the two straight lines shall never meet. And the opposite Proposition might in like manner have been put in the form of declaring, that if of the alternate angles *one is less than the other*, the two straight lines shall meet on the side on which is the alternate angle that is smallest. But the form given by Euclid is the best adapted for future use; which was doubtless the reason why he chose it.

## PROPOSITION *D*.

THEOREM.—*If two straight lines (in the same plane) are cut by a third, and the two interior angles on the same side of the cutting straight line are not together equal to two right angles, but less on one side and greater on the other; the two first-mentioned straight lines, being prolonged, shall at length meet on that side on which are the angles which are together less than two right angles.*

*First Case;* if the two interior angles are each less than a right angle, and equal to one another; as HGB, GHD below. On the straight line HD and at the point H in it,




describe a rectilinear figure DHIK, having its sides and angles respectively equal to those of DHGB; and in the same manner on GB, at G, a rectilinear figure BGLM. Because IHGL is a* rectilinear series, being continued it will† rejoin itself, and the angular points of the series will lie in the circumference of a circle whose radius GO is limited and can be found, and the straight lines drawn from the centre of such circle to the angular points will bisect the angles of the series; that is to say, the straight lines GB and HD, which bisect the angles of the series, being prolonged will meet in O.

* Prop. *A*. Nom. 2.
† Prop. *C*.

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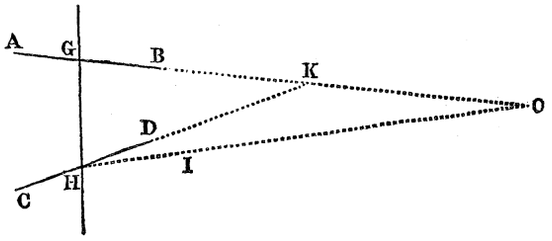
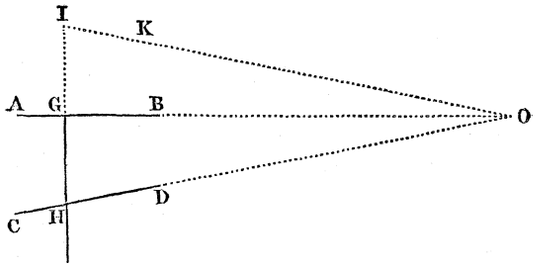
*Second Case;* if the two interior angles are each less than a right angle, but not equal to one another; as HGB, GHD below. Of the two, let HGB be the greater; and at H make




the angle GHI equal to HGB. Because the angles HGB, GHI are each less than a right angle, and equal to one another; (by the First Case) being prolonged they will meet. Let them meet in O. Prolong now HD, and because it cannot meet the perimeter of the triangle GHO in HG or in HO (for then two straight lines would inclose a space), it will meet it in GO, as in K. That is to say, GB and HD being prolonged meet in K.

*Third Case;* if of the two interior angles one is a right angle and the other less than a right angle; as HGB, GHD below, of which HGB is the right angle. In the prolongation of HG take




GI equal to GH, and at I make the angle GIK equal to GHD. Because the angles HIK, IHD are each less than a right angle, and equal to one another; (by the First Case) being prolonged they will meet. Let them meet in O; and join OG. Because in the triangle IHO the angle HIO is equal to the angle IHO, the side HO is equal to IO; and because in the triangles IGO, HGO, the sides IO, IG are equal to the sides HO, HG respectively, and the third side GO is common, the triangles are equal, and the angle IGO is equal to the angle HGO, and because they are adjacent angles they are right angles. But HGB is* also a right angle; therefore GB coincides in direction with GO, and being prolonged will pass through O. That is to say, GB and HD being prolonged meet in O.

* Hyp.

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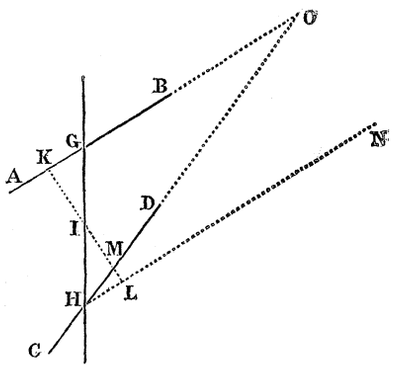


*Fourth Case;* if of the two interior angles which are together less than two right angles, one is greater than a right angle and the other less; as HGB, GHD opposite, of which HGB is greater than a right angle. At H (where is the angle GHD which is less than a right angle) make the angle GHN equal to HGA, or such that HGB and GHN are together equal to two right angles; and because HGB
* Hyp. and GHD are* together less than two right angles, the angle GHD is less than GHN, or HN falls outside of HD. Bisect now GH in I, and from I draw IK perpendicular to GA, and IL
† Hyp. to HN. Because IGA is† less than a right angle, IK will fall on the side of G which is towards A; and for a like reason IL will fall on the side of H which is towards N. In the triangles GIK, HIL, because the angles IGK, IKG are equal to IHL, ILH
‡ Constr. respectively, and the side IG equal‡ to IH, the triangles are equal, and the angle GIK is equal to HIL; and because HIL and GIL are together equal to two right angles, GIK and GIL are together equal to two right angles, therefore KIL is one straight
* Constr. line. And because in the triangle HLM the angle HLM is* a right angle, LMH is less than a right angle, and KMD which is the vertical angle is less than a right angle. Hence of the two interior angles MKB and KMD, one is a right angle and the other less than a right angle. Therefore (by the Third Case) KB and MD (or GB and HD) being prolonged will meet.

In all the possible Cases therefore, it has been shown that if the two interior angles HGB, GHD are together less than two right angles, GB and HD being prolonged will meet. And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, if two straight lines (in the same plane) are cut by a third, and the two interior angles on one side of the cutting straight line are not together equal to two right angles, but less on one side and greater on the other; the two first-mentioned straight lines, being prolonged, shall at length meet on that side on which are the angles which are together less than two right angles. Which was to be demonstrated; *and is Euclid's Axiom.*

END OF THEORY OF PARALLELS.

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CHAPTER I.—*On the various other ways in which the spiral might be applied to the establishment of the doctrine of parallels.*

1. It might be applied to prove that in the quadrilateral figure formed by drawing from the ends of a straight line two straight lines equal to one another towards the same front, perpendicular to the first side or base, and joining their extremities, the angles opposite to the base are not *less* than right angles. For if they were less than right angles, then by placing a number (increasable at discretion) of fac-simile figures of the kind in question side by side so that their bases should form one straight line, there must be presented a rectilinear series which being continued ever so far would never meet the straight line of the bases. But this is impossible; for an equiangular spiral might be applied as in Prop. *B*, to which a rectilinear series having the angle of the series equal to the sum of two of the angles of the quadrilateral figure which are supposed less than right angles, should necessarily be interior, and because the spiral would cut the radiant within a limited distance after it had revolved through an angle equal to half the angle of the series, the series would. There cannot, therefore, be a series as supposed; that is to say, there cannot be quadrilateral figures as above, having the angles opposite to the base less than right angles. Which usefully displays what the necessity is for confining (or as it might be called, bridling) the rectilinear series with the spiral at all. For without it, there might here be an example of a rectilinear series, in which there would never cease to be another of the equal straight lines presented to be cut by the radiant, yet the series would not continue to cut the radiant after the radiant had revolved through an angle equal to half the angle of the series.

2. After establishing the above, it would be practicable either to proceed to show that the same angles cannot be *greater* than right angles (as was done in the First Edition of "Geometry without Axioms" by placing a number at discretion of fac-simile figures of the kind in question side by side so that their bases form one straight line as before, and proving that the sides opposite to the bases, being prolonged towards the same hand, must cut the perpendicular side of the first figure at perpetually increasing distances from its extremity, and consequently after a certain number of figures, the side opposite to the base of one of them, being prolonged, must cut the straight line in which are all the bases; which is impossible, for those straight lines are parallel); whence may be concluded that the angles opposite to the base are *equal* to right angles, and that consequently the angles of every right-angled triangle, and subsequently of every triangle whatever, are together equal to two right angles;—

3. Or it might be inferred at once, that the angles of any right-angled triangle, and subsequently of every triangle whatever, cannot be together *less* than two right angles; whence it follows that in any triangle the two interior and opposite angles cannot be together less than the exterior angle; from which may be proved Euclid's Axiom, as was shown in the Fifth Edition of "Geometry without Axioms."

4. Or instead of all this, it might be demonstrated that through any three points not in the same straight line, a circle of limited radius may be described. For if on joining the points, the sides about the angle which is either the greatest or not less than any other in the triangle, are not equal, add to the shorter till they are;

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and because about this isoskeles triangle, as being part of a rectilinear series, a circle of limited radius may be described, the perpendiculars which bisect the equal sides will meet in the centre of such circle. Whereupon can be shown that the perpendicular which bisects the original smaller side, will also meet the perpendicular which bisects the greater; by which the centre of the circle which will pass through the three points given, will be found.

5. Or the spiral seems capable of being applied to prove, that if two straight lines of unlimited length both ways, are perpendicular to a third, they shall be everywhere equidistant. For if not, let a circle be described having for its radius the perpendicular which is between those two straight lines, and let this circle roll along one of them. And if the other is not always equidistant, the centre of the circle must describe a line either interior to it, or exterior; as, first, let it be interior. Upon which might be shown, from the equality of the parts respectively concerned, that the line described must be a curve alike in all its portions, and consequently such that if two points in it were joined, the angles made by the curve with the joining line at its two extremities must always be equal; whence a spiral might be applied to which the curve must always be interior, and because the spiral would cut the radiant within a limited distance after it had revolved through an angle sufficient to make it meet the straight line on which the circle rolls, the curve must; which is impossible, for the straight line joining any two points in the curve must always be parallel to the straight line on which the circle rolls. The two straight lines therefore cannot increase their distance. And in a similar manner might be shown, that if the line described by the centre of the rolling circle was exterior, or the distance of the two straight lines from one another decreased, a curve would be formed whose radiant, being prolonged backwards, would at some time meet the straight line on which the circle rolls; which is impossible as before. The two straight lines therefore can neither increase their distance nor diminish it; that is to say, they will always be equidistant. From which it is easy to infer, either the equality of the three angles of every triangle to two right angles, or Euclid's Axiom.

6. Lastly, without naming a spiral at all, by making a straight line revolve after the manner of a radiant and pass over the equal straight lines which compose a rectilinear series, it might be demonstrated that the velocity *fg* (See the Second figure in Prop. *B*) with which the travelling point is moving along the radiant at different points *e*, is to the velocity *ef* with which *e* is moving in the transverse direction (the revolution of the radiant being supposed uniform), in ratios such that *fg* is always smaller in respect of *ef* than, for instance, *qp* is of C*q*; whence, since even if they were in the constant ratio of *qp* to C*q*, the velocities along the radiant and the distances would be mutually limited, still more will they be so when instead of the velocities along the radiant being in that constant ratio to the velocities in the transverse direction, they are continually below it. After which the length of the series may be proved to be limited, in the same way as the length of the spiral in Prop. *A*, Cor. 4. And this to the practised mathematician may appear the most direct mode of proof, and the freest from anything that can be removed. But the perception of the distance being limited in the case of the spiral, is the way in which the truth most naturally emerges, and will probably always be the most impressive.

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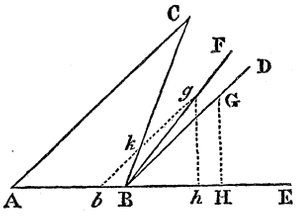
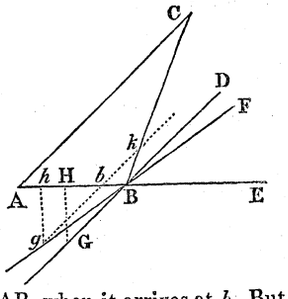
CHAP. II.—*On some Propositions capable of being demonstrated independently of the doctrine of parallels.*

1. It can be demonstrated, that whether the three angles of every triangle are together equal to two right angles or not; if the sides of any triangle are diminished without limit assigned, two of the angles continuing always the same, the sum of its angles is ultimately equal to two right angles. Or in other words, there is no amount less than two right angles, than which the angles of the triangle, on the sides being diminished at discretion, cannot be shown to be greater; and no amount greater than two right angles, than which they cannot be shown to be less. As, let ABC be a triangle of any kind, and at B draw BD making with BE an angle DBE equal to the angle A; and let a straight line move from A along AB, making with AB always an angle equal to the angle A, till it comes into the situation BD. If then any angle be taken less than CBD by ever so small a difference DBF, the moving straight line before arriving at the situation BD, shall make a triangle whose third angle is greater than CBF. For from any two points in BD and BF draw perpendiculars to AE; and if these perpendiculars are not equal, let the smaller of them be moved along AE in the direction that may be required, keeping always at right angles to it, till it meets the other of the straight lines BD or BF which had the greater perpendicular. In this way may always be found two equal perpendiculars, as HG, *hg*. Take *hb* equal to HB, and join *gb*. By reason of the equality of two sides respectively and of the included angle, the triangle *ghb* will be equal to the triangle GHB, and the angle *gbh* to the angle GBH; that is to say, *bg* will be the situation of the straight line which moves along AB, when it arrives at *b*. But the angle *bk*B is greater than *k*B*g* or CBF, because it is the exterior angle of a triangle B*kg*, and the other is one of the interior and opposite. And for the like reason, any triangle made by the moving straight line as it approaches still nearer to B, will have its third angle greater than CBF. So on the other hand, if any angle be taken greater than CBD by ever so small a difference DBF, let HG and *hg* be any two equal perpendiculars to AE as before, meeting the prolongations of DB and FB; take *hb* equal to HB, and join *gb*, prolonging it to an unlimited extent on the side of *h*. Whereupon may be shown as before, that the angle *gbh* will be equal to the angle GBH; and because the angle GBH is equal to the angle A, and the angle *gbh* to *kb*B, the angle *kb*B is equal to the angle A, and *bk* will be the situation of the straight line which moves along AB, when it arrives at *b*. But the angle *bk*B (or *gk*B) is less than the angle CBF, because this last is the exterior angle of a triangle B*kg*, and the other is one of the interior and opposite. And for the like reason any triangle made by the moving straight line as it approaches still nearer to B, will have its third angle less than CBF. Hence no difference can be assigned so

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some kind at the centre, and this angle would be contained some number of times and no more in the sum of four right angles, the same number of yards and no more would be in the orbit.

### PROPOSITION *A*. NOMENCLATURE 1.

Properties demonstrable after the establishment of the doctrine of parallels, have caused this spiral to be called the *logarithmic*. But there would have been a manifest impropriety in introducing it by that title here.

### PROPOSITION *A*. NOMENCLATURE 2.

The term *polygonic* used in a former publication, has been avoided; because though in strictness it only means *having many angles*, it is apt to suggest the idea of the self-rejoining figure known by the title of *polygon*, which is directly contrary to the intention here, the question whether the series will ever rejoin itself being in reality the great matter in dispute.

### PROPOSITION *B*.

The necessity for proving the intercepted portion of the radiant to be limited, and the possibility which would otherwise exist for a rectilinear series not to continue to intersect the radiant after the radiant had revolved through an angle equal to half the angle of the series, are illustrated in Chap. I. § 1. of the Appendix; which see.

### PROPOSITION *C*.

The number of the equal straight lines to be added on the further side of X to make the series rejoin itself, will be the same as were on the hither side, or one more. If X passes through an angular point as G, or divides one of the equal straight lines as GH into two equal parts, the number required will be the same; as may be seen by supposing the figures on different sides of AX to be applied together by doubling about AX. If X divides GH into two unequal parts, and XH is the greater, the number will still be the same; but if XH is the smaller, then one more.

### APPENDIX. Chap. II. § 3.

As, for example, if an objection were raised upon making the angle *b*A*d* (in Prop. *A*) equal to the angle BAD, as being what would be passed through in equal times by a uniform revolution of the radiant; and then urging that nothing is known respecting the equality of the angles at D and *d*, in triangles whose sides about the right angle are proportional. To which one answer would be, that the angle at *d* could not be greater than the angle at D; and if it were less (or the spiral constructed proved to be one in which the angle between the radiant and the curve decreased), no impediment would thence arise to the use intended to be made of the spiral.

But besides this, there would be the other answer,—that producing what *will not* prove a given proposition, is no argument against what *will*.

THE END.

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